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INVESTMENT AND FINANCIAL POLICIES OF A PUBLICLY OWNED FIRM

A. Anastasopoulos

DP 9109

Concordia
UNIVERSITY

1455 de Maisonneuve W. Montreal Quebec H3G 1M8

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ABSTRACT

The determination of the real and financial decisions of a publicly owned firm is studied in a neoclassical framework with irreversible investment and uncertainty about future prices and interest rates. The main assumptions are (i) the management knows the equilibrium price of capital and (ii) it maximizes the expected utility of dividends-per-share.

It is shown that: investment follow a deterministic rule, capital gains exists only if net capital formation takes place, and market value maximization follows as a consequence of optimal policies. Stock buy-outs occur whenever there are not attractive real investment opportunities. In addition, the if the policy function is homothetic, the issue of new equity depends linearly on the stock of outstanding shares, while borrowing is a linear function of the capital stock.

I. Introduction

The real and financial policies of a publicly owned firm has been the subject of long debate in the theory of finance. Modigliani and Miller (M-M) ([1958], [1963]) have established two robust propositions: In perfect capital markets optimal investment policies are independent of how investment are financed, and the market value of a firm is independent of its dividend policies. Thus, the financial structure of the firm is irrelevant, and the development of any meaningful theory of corporate finance a hopeless task. The modern theory of finance has emerged from efforts to elude the implications of the M-M theory by weakening its assumptions. To do so researchers have consider the role of transaction and monitoring costs ([Shleifer and Vishny [1986]), taxes (Miller [1977]), market imperfections, asymmetric information and signaling (Ross [1977a], Lealand and Pyle [1977], Miller and Rock [1985]), agency costs (Wilson [1968], Ross [1973], Myers [1977]), etc. This paper follows the same route. Before we specify our approach, we discuss an aspect of the M-M theory which is relevant for our purposes.

The Modigliani-Miller theory is founded on the neoclassical paradigm which assumes that the management acts on behalf of all shareholders who exercise the ultimate control.[1] To be consistent with this objective, the management should maximize either the expected utility of profits or the market value of the firm. Expected utility is ambiguous; shareholders may have diverse attitudes to risk and it is not clear whose risk preferences are relevant. The market value criterion also is restrictive; it is satisfactory only under perfect competition and complete markets. In this paper the management is assumed to maximize the expected utility of dividends-per-share. This objective, which renders the financial structure of the firm determinate, can be justified on several grounds. One justification, which is not along the lines of the neoclassical paradigm, is that the management caters to the needs of a group of shareholders who own collectively one share at all times.[2] Alternatively, the attitude to risk which the management displays is policy decision ; it is a signal which intends to maximize the flow of investment funds to the firm. Below we elaborate on this view.

In real life firms engage in public relations which create in the eyes of the public a certain image about the firm. One could interpret these activities as signals which intend to inform the public about the degree of risk-taking which the firm intends to display in making investment.[3] By doing so they differentiate their shares from those of other firms which are more or less risk-takers, and they appeal for investment funds to investors of certain type. Namely, they appeal to those who would chose an optimal portfolio with

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[1] A publicly owned firm a legal person. As such it has the ability to own property, make contracts, undertake obligations, etc. Because it lacks the capabilities of natural persons, its interests are looked after by the management. The management, in turn, is supervised by the shareholders, ie. a body consisting of those who have contributed to the present assets of the firm. Stressing different aspects of these facts results in different theories about the nature of the firm and the role of the management.

[2] This arrangement is common among public firms which evolved out of family-owned companies. Regulations about insider trading justify the assumption that the controlling group owns one share at all times.

[3] An telling example, is the announcement of key appointments.

risk-return characteristics similar to those of the firm's shares.[4] In addition, by maximizing dividends-per-share the management caters to the needs of the "average" shareholder in this group. This interpretation has interesting implications in a general equilibrium framework, but it is not perused further here.

The determination of the real and financial decisions of a publicly owned firm (public firm) is studied in a dynamic setting with irreversible investment and uncertainty about future prices and interest rates. More specifically, firms transform output into capital, and capital goods are not transferable among firms. The ownership of the capital assets is traded through shares. Thus, one should not observe a price for capital as such. However, the price of capital can be imputed from initial stocks, the price of financial assets, and the optimal policies of the firms. Here, we suppose that each firm knows the equilibrium price of capital and it takes it to be beyond its control. This assumption leads to a deterministic rule of investment, and it preserves the notion of perfect competition when capital is not traded directly. At the end of Section 2 we indicate how the price of capital and of the financial assets could be determined in a Walrasian market.

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[4] It is supposed that these individuals would prefer to invest in one firm rather that many, especially if there are transaction and monitoring costs. Furthermore, under uncertainty (rather that risk) and asymmetric information, it is reasonable to suppose that investors are more likely to to invest in a firm when they are confident as to how the management intends to deal with unforeseen contingencies and with privileged information. In this view firms endeavour to create a niche in the market for funds by adopting the appropriate risk preferences.

The process of capital formation which we have assumed and the view that public firms compete for funds, relates to another topic in the theory of the firm; namely the returns to entrepreneurial skills. The importance of the factor "entrepreneur" is readily recognized, however traditional theory does not treat its reward adequately. It has been suggested by L. McKenzie that entrepreneurs receive economic rents: under decreasing returns to scale they receive what is left after capital and labour are paid. In this scheme the entrepreneurial factor receives the value of its marginal product only if a production function which incorporates all three factors displays constant returns. If this function displays increasing (decreasing) returns, entrepreneurs are not compensated optimally (are over-compensated); this follows from Euler's theorem. It has been recognized that entrepreneurship yields externalities which benefit other firms as well; thus increasing returns seem appropriate. But then, this factor does not receive the value of its marginal product and, therefore, it is not optimally supplied in the long-run.

An alternative to the approach of McKenzie can be based on the "Penrose effect" of Uzawa [1969].[5] Uzawa, in order to develop an index of real capital, assumed that capital assets incorporate the entrepreneurial and managerial skills of the firm. This suggests that the manager-entrepreneur may be seen as an agent who designs risky ventures and in order to finance them he is compelled to auction in advance to investors the ownership of a

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[5] These views on the role of the management appear to be relevant for a significant portion of real-life public firms.

part of the assets which will be created. The designing of such ventures requires considerable expertise and, in doing so, the entrepreneur acquires over time firm-specific skills. Thus, he is compelled to follow policies which do not lead to bankruptcy, and at the same time to discriminate against numerous populations of future shareholders.[6] In this view, the price of capital in each period reflects the the value which the market places on the skills of the entrepreneur. Indeed, as it will be seen later, capital gains accrue to the owners of the existing capital assets only to the extend that the price of capital is sufficiently high to warrant net capital formation.[7] The role of the management and its objectives have been discussed in the literature under the topic of "agency costs" What is stressed here is that agency costs may be negative when it comes to entrepreneurial skills.

As it will be seen later, the approach taken here makes the problem of the public firm formally analogous to that of a dynamic portfolio. Thus, the optimal policies of the firm may be formally treated with techniques already developed for such problems. In Section 2 the model is presented and the conditions which must be satisfied by the real and financial policies of the firm are discussed. There, it is also indicated how the equilibrium price of capital and shares might be determined in a Walrasian market. Section 3 contains the assumptions which are required to make the problem tractable, and

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[6] This is achieved by maximizing dividends-per-share.

[7] In our model the price of capital may be common to all firms. In a model where this price differs among firms, we conjecture that successful entrepreneur (and the owners of outstanding shares) capture in advance the externalities generated by the entrepreneurial activity.

some preliminary results. In Section 4 the optimal policies are derived and their properties are discussed. It will be shown that investment follow a deterministic rule, capital gains exists only if net capital formation takes place, and market value maximization follows as a consequence of optimal policies. Finally, the if the policy function is homothetic, the issue of new equity depends linearly on the stock of outstanding shares, while borrowing is a linear function of the capital stock.

II. The Model

2.1 The Environment of the Firm

We consider a one commodity economy, say corn. Corn is produced by a number of competitive firms which are owned by individual investors, the shareholders. The output of the period is either consumed within the period or transformed into new capital goods.

At the beginning of a period each firm owns a stock of firm-specific capital goods, and hires labour services in order to produce corn. Furthermore, it transforms corn into new capital goods which become available at the end of the period and it distributes dividends to its shareholders. Capital formation is financed through a combination of retained earnings, debt finance, and equity finance.[8]

Debt obligations mature at the end of the period. Equities, which are

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[8] These sources of finance are identified, respectively, as corn either produced by the firm, or borrowed from the public, or obtained in exchange for the ownership of a part of the firm's capital assets.

titles of ownership to the capital assets of the existing firms, impose linear sharing rules. Share splits are not permitted. Firms may not own shares issued by other firms but they may repurchase their own outstanding shares. A firm may be sold as a unit, in which case it ceases to exist and it is transformed into a new institution.

The activities of the economic agents are coordinated in a Walrasian market which operates at the beginning of each period. Commodities traded in the market are: corn, labor services, bonds, equities, and indirectly through equities, the capital assets of the existing firms. All prices and the interest rate are denoted in terms of corn. In equilibrium the spot price of all commodities and the price of new capital are known with certainty. Future prices and interest rates are random. The process of attaining equilibrium is stated explicitly at the end of Section 2.3.

2.2 Notation

p_t: the forward price in period t of one unit of capital available in t+1.

$\hat{p}_t$: the value of a unit of outstanding capital in t.

x_t: dividends per share in t.

$\hat{q}_t \equiv q_t + x_t$: the (derived) price of an outstanding share in t.

q_t: the (derived) price of a new share issued in t.

$\tilde{r}_t$: one plus the nominal interest rate of period t.

r_{t+1}: one plus the (random) real rate of return on bonds issued in t; i.e. $\tilde{r}_t$ divided by the anticipated rate of inflation between period t and t+1.

y_t: the output of corn during t.

w_t: the real wage rate of t.

$\rho_t \equiv \rho(w_t)$: the marginal physical product of capital in t when the firm adjusts labor optimally.

N_t: Labour employed during period t.

K_t: the stock of capital at the beginning of t.

I_t: the amount of corn transformed into capital by the firm during t.

$z_t \equiv K_t\gamma$: the gross addition to the capital assets of the firm during t.

γ_t: the rate of gross capital formation during t.

B_t: the number of bonds outstanding at the end of t.

S_t: the number of shares outstanding at the beginning of t.

E_t: the number of new shares issued in t.

$\frac{\hat{p}_{t+1}}{p_t}$: one plus the (random) rate of returns of one unit of future capital between period t and t+1.

$\alpha_t \equiv \frac{B_t}{(\hat{q}_t - x_t)(S_t+E_t)}$: the debt-equity ratio of the firm in t.

$h_t \equiv (\rho_t, p_t, r_t)$: the "data" of period t.

$H_t \equiv \{h_j,\ j \le t\}$: the data up to and including period t.

2.3 Real and Financial Policies

The financial decisions of the firm are assumed to satisfy the following equilibrium conditions: At the opening of the market the value of the capital assets of the firm is defined to be equal to the value of its outstanding debt and equity:

(2.1) $V(K_t) = \hat{q}_t S_t + B_{t-1} r_t$.

For the same reasons, the value of its capital the value of its capital assets at the end of the closing of the market is:

(2.2) $p_t K_{t+1} = q_t S_{t+1} + B_t = (\hat{q}_t - x_t)(S_t+E_t) + B_t$,

where, $K_{t+1} = K_t(1+\gamma_t-\delta)$, and $\hat{q}_t - x_t = q_t$.

In addition, the sources of funds are constrained to be equal to their uses:

$$y_t + q_t E_t + B_t = w_t N_t + I_t + S_t x_t + B_{t-1} r_t.$$

Concerning the "real" decisions it is assumed that the firm operates in a competitive environment and its production is subject to constant returns. Hence, its employment policies can be "maxed-out", and the economic rent to capital is equal to its marginal physical product:

$$y_t - w_t N_t = K_t \rho(w_t) \equiv K_t \rho_t .$$

In addition to producing corn, the firm transforms corn into new capital goods which become available at the end of the period. This process is expressed in the following assumption:

Assumption 1: $$\frac{I_t}{K_t} = g\left(\frac{z_t}{K_t}\right) = g(\gamma_t); \quad g(0)=0,\ g'>0,\ g''>0,\ g'(0)=1.$$

Assumption 1 is a two-commodity version of the Penrose effect, [Uzawa, 1969].[9]

Combining the conditions which are satisfied by the "real" decisions with the "sources and uses of funds" and with constraint and (2.2) we obtain:

$$(2.3) \quad \hat{q}_t S_t + B_{t-1} r_t = K_t\left[\rho_t - g(\gamma_t) + p_t(1+\gamma_t-\delta)\right].$$

Expressions (2.3) and (2.1) yield:

$$(2.4) \quad V(K_t) = K_t\left[\rho_t - g(\gamma_t) + p_t(1+\gamma_t-\delta)\right] \equiv K_t \hat{p}_t .$$

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9 In Uzawa new capital goods incorporate the organizational abilities and the technical skills of the management. Here the investment process is given a two-commodity interpretation: corn is transformed into firm-specific capital goods with the help of the existing capital stock. If g differs among firms it is because some firms are more efficient than others in creating capital. This is, presumably, because efficient firms possess superior managerial skills and other intangible factors. With this modification the Uzawa approach to investment is well suited for studying problems of equity finance and the role of the entrepreneur.

Thus, the value of a unit of existing capital consists of (a) its economic rent in producing corn, and (b) market value (net of reproduction costs) of its contribution to the future capital stock. It is suggestive to rewrite (2.4) as:

$$(2.4)' \quad \hat{p}_t \equiv [\rho_t - g(\delta) + p_t] + [p_t(\gamma_t-\delta) - \{g(\gamma_t) - g(\delta)\}].$$

The first term in brackets is the net (of reproduction costs) returns plus the forward price of an existing unit of capital. If $\hat{p}_t$ captures all the future returns to capital, the second term is a surplus value, or capital gains.

Finally, leading (2.1) and (2.4) by one period and combining the resulting expression with (2.2) we obtain the recursive relation:

$$(2.5) \quad S_{t+1}\hat{q}_{t+1} = (\hat{q}_t - x_t)(S_t + E_t)\frac{\hat{p}_{t+1}}{P_t} + \left(\frac{\hat{p}_{t+1}}{p_t} - r_{t+1}\right)B_t ,$$

where,

$$(2.6) \quad \hat{p}_{t+1} = \rho_{t+1} - g(\gamma_{t+1}) + p_{t+1}(1+\gamma_{t+1}-\delta),$$

from (2.2), and

$$(2.7) \quad S_t \hat{q}_t = K_t \hat{p}_t - B_{t-1} r_t .$$

From (2.5) and (2.6) is seen that the rate of return from investing in one unit of future capital is a function of the rate of gross capital formation.

It is assumed that the management follows a feasible policy (x_t, z_t, E_t, B_t) which maximizes $E\left[\sum_{t=0}^{\infty} \alpha^t u(x_t)/H_o\right]$ subject to (2.5), K_o, S_o equal to some constant, and $B_{-1} = 0$. The expression $E\left[\cdot / H_t\right]$ is the expectations operator conditioned on present and past information.

It was mentioned above the consumption-saving plans of individuals and the production-investment plans of firms are coordinated in a Walrasian market where m firms operate. The sequence of events in the market session of period t is as follows: Each firm possesses a certain amount of capital, and each individual owns (as a result of past decisions) an endowment of outstanding bonds and shares of the various firms. The auctioneer announces a vector of "primary" prices (denoted in terms of corn); i.e. prices which are sufficient for the firms to determine their optimal plans. These prices are: $(w_t, \tilde{r}_t, p_{1t}, p_{2t}, \ldots, p_{mt})$, where the first subscript of p refers to a firm. Given these prices and the expectations of the management about the future real rate of interest, the jth firm determines its optimal production, investment, and financial decisions $(y_{jt}, N_{jt}, x_{jt}, z_{jt}, E_{jt}, B_{jt})$, as shown in Theorem 1 and Corollary 1 below. In addition, it determines a vector of "derived" prices. Derived prices are asset prices computed from the primary prices, the optimal decisions of the firm, and the conditions which must hold if equilibrium prevails in the demand and the supply of shares; here from conditions (2.2), (2.4), and (2.7). Primary and derived prices are communicated to individuals individuals who, in turn, plan their optimal labour supply, consumption, saving, and portfolios of equities and bonds for the period. The auctioneer tallies the results and compares the total demand and supply of each commodity. If they match primary and derived prices are declared equilibrium prices, if not new primary price are announced...etc. In the paper we do not deal with the existence of equilibrium, we assume that the the management of the representative firm forms expectations about the future equilibrium prices and we concentrate on the optimal production and financial plans given these expectations. The proofs of the main results are found in the Appendix.

III. Assumptions - Preliminary Results

To make the problem tractable we make the following additional assumptions and definitions.

Assumption 2: $u(x_t)$ is strictly concave and non-decreasing for $x_t \geq 0$, and $u'(0) = \infty$ for t=0,1,2... .

Definition 1: A_t is the set of (ρ_t, p_t, r_t), t=0,1,2...such that $\Pr\left\{(\rho_t, p_t, r_t) \in A_t\right\} = 1$.

Assumption 3: (ρ_t, r_t, p_t) in A_t, t=0,1,2,..., are random variables with finite supports which evolve according to a Markov process. Their supports satisfy, respectively, the following conditions:

(i) $0 < \bar{\rho}_t < \bar{\bar{\rho}}_t \leq 1$; $g(\delta) < \bar{\rho}_t$.

(ii) $1 \leq \bar{r}_t < \bar{\bar{r}}_t < \infty$.

(iii) $1 \leq \bar{p}_t < \bar{\bar{p}}_t < \infty$.

Assumption 4: For any realization of p_t, t=0,1,2,....

$$\bar{r}_{t+1} < \frac{1}{p_t}\left[\bar{\bar{\rho}}_{t+1} + \bar{\bar{p}}_{t+1}(1-\delta)\right], \text{ and } \frac{1}{p_t}\left[\bar{\rho}_{t+1} - g(\delta)+1\right] < \bar{\bar{r}}_{t+1}.$$

Definition 2: $Z(p_t), p_t \in A_t$, t=0,1,2,..., is the set of investment policies γ_t which satisfy the condition $p_t z_t \geq I_t = K_t g\left(\frac{z_t}{K_t}\right)$; i.e. $p_t \gamma_t \geq g(\gamma_t)$.

Assumption 5: γ_t is an admissible policy if $\gamma_t \in Z(p_t) \cup [0,\delta]$, where $[0,\delta]$ is a proper subset of $Z(p_t)$.

Assumption 6: $S_t \geq \bar{\varepsilon}_t$, $\bar{\varepsilon}_t > 0$, for a su

fficiently small $\bar{\varepsilon}_t$.

Assumption 7: Equity rules require that $p_t K_{t+1} - B_t \geq 0$, t=0,1,2,... .

Assumption 8: $K_t = 0$ for some finite t implies $\hat{q}_{t+k} = x_{t+k} = 0$, k=1,2,...

Assumption 2 commits the management to exhibit risk aversion in its decisions. The second part of Assumption 3(i) requires that the marginal product capital exceeds its replacement cost. Assumption 4 states that the returns to a unit of future capital are not expected to be always greater, or less than, the return on bonds during the same period. Assumption 5 defines the set of admissible policies; policies which lead to net capital formation are admissible if the market value of new capital exceeds its cost of production. Assumption 6 prohibits firms from buying back all of their shares. Assumption 7 is the no-bankruptcy condition; here it is justified because of the stress on the entrepreneurial aspects of management. Assumption 8 allows for the possibility that the firm may be sold as a unit, in which case its old shareholders loose any claims to future dividends which the firm may generate under the new administration.[10]

The following preliminary results are established.

Lemma 1: $Z(p_t)$ is compact, and its greatest lower bound is zero.

The validity of the statement follows immediately from Assumptions 1, 3(i), and 7.

Lemma 2: $\hat{p}_t > 1$ for all $(\rho_t, p_t, r_t) \in A_t$, $t=0,1,2,\ldots$.

The proof follows from (2.4) and Lemma 1: $\hat{p}_t \geq \bar{\rho}_t - g(\delta)+1$, which is strictly positive by Assumption 3.

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[10] The depreciation process guarantees that $K_t > 0$ for all finite t's. Assumption 8 Makes it possible for the management to sell the capital assets of the firm in order to avoid bankruptcy.

Lemma 3: $\Pr\left\{\frac{\hat{p}_{t+1}}{p_t} - r_{t+1} \left\{\begin{matrix}\geq \\ <\end{matrix}\right\} 0/H_o\right\} > 0$ for $(\rho_{t+1}, p_{t+1}, r_{t+1}) \in A_{t+1}$ and,

$\gamma_{t+1} \in Z(p_{t+1}) \cup [0,\delta]$, $t=0,1,2,\ldots$.

Thus, investing in future capital may lead to either profits or losses with positive probability. The proof is given in the Appendix.

Assumptions 1-8, Lemmas 1-3 and (2.5)-(2.7) define the set of feasible policies outlined in Definition 3.

Definition 3: $(x_t, \gamma_t, E_t, B_t)$, $t=0,1,2,\ldots$ is a feasible policy if it satisfies the conditions

(3.1) $\Pr\left\{(\hat{q}_t - x_t)\frac{\hat{p}_{t+1}}{p_t}(S_t + E_t) + \left(\frac{\hat{p}_{t+1}}{p_t} - r_{t+1}\right)B_t \geq 0/H_t\right\} = 1.$

(3.2) $\gamma_t \in Z(p_t) \cup [0,\delta]$.

(3.3) $E_t \geq \bar{\varepsilon}_t - S_t$, $x_t \geq 0$.

(3.4) K_o, S_o = some positive constants, $B_{-1} = 0$,

where,

(3.5) $\hat{p}_t = \rho_t - g(\gamma_t) + p_t(1+\gamma_t-\delta)$, and

(3.6) $S_t\hat{q}_t = K_t\hat{p}_t - B_{t-1}r_t$.

It can be seen that the set of feasible solutions is not empty. The policy $\gamma_t = \delta$, $B_t = 0$, $E_t = 0$, $S_o X_t = K_o[\rho_t - g(\delta)]$ for all t, satisfies (3.2) and (3.3). Furthermore, when combined with (2.1) yields $S_t = S_o\hat{q}_t = K_o\hat{\rho}_t = K_o[\rho_t - g(\delta) - p_t] > 0$, by Assumption 4, and $S_o(\hat{q}_t - x_t) = K_o\rho_t$. Thus, it satisfies (3.1) as well.

The optimization problem of the firm is to choose policies $(x_t, \gamma_t, E_t, B_t)$,

$t=0,1,2,\ldots$, which

(3.7) maximize $E\left[\Sigma\alpha^t u(x_t)/H_o\right]$

subject conditions (3.1),(3.2),(3.3), (3.5), and (3.6),

where (x_t,γ_t,E_t,B_t) are the controls, and (K_t,S_t) the state variables.

IV. Optimality Conditions

To proceed, the optimization problem of the previous section is modified into one where the debt-equity ratio appears as one of the control variables. We start with the following Lemma.

Lemma 4: If $u(x_t)$ is bounded above, the optimum x_t, $t=0,1,2\ldots$, is such that $0<x_t<\hat{q}_t$. If $u(x_t)$ is not bounded above but bounded below, the optimum x_t, $t=0,1,2,\ldots$, is such the $x_t<\hat{q}_t$. In the second case, if the policy $x_t=0$ for infinitely many t's makes the $\lim_{t\to\infty} a^t E[u(\hat{q}_t)/H_o]\equiv\infty$, then this policy is optimal.

The proof of Lemma 4 is given in the Appendix.[11] As a result of Lemma 4 the optimization problem of Section 3 can be reformulated as follows:

(4.1) maximize $E\left[\Sigma\alpha^t u(x_t)/H_o\right]$

(4.2) subject to $Pr\left\{(\hat{q}_t-x_t)\left[\frac{\hat{P}_{t+1}}{P_t}+\left(\frac{\hat{P}_{t+1}}{P_t}-r_{t+1}\right)\alpha_t\right]\geq 0/H_t\right\}=1$

(4.3) $Z_t\in Z(p_t)\cup[0,\delta]$

(4.4) $x_t\geq 0$, $\hat{q}_0=\frac{K_0}{S}\hat{p}_0$, K_o,S = some positive constants, $B_{-1}=0$,

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[11] Lemma 4 and its proof are adopted with minor modifications from Lemma 1 of Anastasopoulos A. and Kounias S. (1975).

where,

(4.5) $\hat{p}_t=\rho_t-g(\gamma_t)+p_t(1+\gamma_t-\delta)$.

The set of α_t's which satisfy (4.2) must meet certain boundary conditions imposed by Assumptions 4 and 6. It can be easily seen that by Assumption 4,

$$(4.6)\quad -1>\left\{\frac{p_t\bar{\bar{r}}_{t+1}}{\bar{\bar{\rho}}_{t+1}-g(\bar{\bar{\gamma}}_{t+1})+\bar{\rho}_{t+1}(1+\bar{\bar{\gamma}}_{t+1}-\delta)}-1\right\}^{-1}\leq\alpha_t\leq\left\{\frac{p_t\bar{\bar{r}}_{t+1}}{\bar{\rho}_{t+1}+g(\delta)+1}-1\right\}^{-1}>0$$

where $\bar{\bar{\gamma}}_{t+1}$ is as defined in the proof of Lemma 3. In addition,

$$(4.7)\quad -1\leq\alpha_t=\frac{p_tK_t(1+\gamma_t-\delta)}{(\hat{q}_t-x_t)(S_t+E_t)}-1\leq\frac{p_tK_t(1+\gamma_t-\delta)}{(\hat{q}_t-x_t)\bar{\varepsilon}_t}-1,$$

from (2.2), Assumption 6, and the definition of α_t. It is assumed that the upper boundary of the set defined by (4.7) can be made larger that the corresponding boundary of (4.6) by the appropriate choice of $\bar{\varepsilon}_t$. Such choice is always possible in view of Lemma 1 and Lemma 4. Thus, conditions (4.6) and (4.7) imply that the set of feasible α_t's which satisfy (4.2) must lie in the interval [12]

$$(4.8)\quad \bar{\alpha}_t\equiv -1\leq\alpha_t=\left\{\frac{p_t\bar{\bar{r}}_{t+1}}{\bar{\rho}+g(\delta)-1}\right\}^{-1}\equiv\bar{\bar{\alpha}}_t.$$

The following theorem gives the optimality conditions of the maximization problem (4.1) - (4.5).

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[12] It is not possible to derive the optimality conditions when the upper boundary of the set of admissible α_t's is equal to the last expression in (4.7)

Theorem 1: Let $(x_t, \gamma_t, E_t, B_t)$ be a feasible policy which satisfies the following conditions:

(i) $p_t = g'(\gamma_t)$

(ii) $u'(x_t) - \alpha E\left[u'(x_{t+1})\left\{\frac{\hat{p}_{t+1}}{p_t} + \left(\frac{\hat{p}_{t+1}}{p_t} - r_{t+1}\right)\alpha_t\right\}/h_t\right] = 0,$

(iii) $E\left[u'(x_{t+1})\left(\frac{\hat{p}_{t+1}}{p_t} - r_{t+1}\right)/h_t\right]\begin{cases} > 0 \text{ for } \alpha_t = \bar{\alpha}_t \\ = 0 \text{ for } \alpha_t \in (\bar{\alpha}_t, \bar{\bar{\alpha}}_t) \\ < 0 \text{ for } \alpha_t = \bar{\bar{\alpha}}_t, \end{cases}$

(iv) $\lim_{t\to\infty} a^{t+1} E\left[u'(x_{t+1})\hat{q}_{t+1}/H_o\right] = 0.$

Then, the policy $(x_t, \gamma_t, E_t, B_t)$ is optimal.

In the Appendix it is shown that if there exists a policy which satisfies the above conditions, this policy overtakes eventually any other policy. The proof does not require that we consider only polices which make the objective function finite. When only such policies are considered, as it is required in Dynamic Programming, it can be seen that the optimality conditions of Theorem 1 are necessary and sufficient for the problem (4.1)-(4.5).

The optimal investment policy is a deterministic rule which maximizes the market value of the capital assets of the firm. The maximization of the market value is not assumed, but derived as a result of optimizing behavior.[13]

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[13] Under certainty and with constant prices and interest rates, we must have $\frac{\hat{p}}{p} = r$. Then, the policy $p = g(\gamma)$ yields $p = \frac{\rho - g(\gamma)}{(r-1)(\gamma-\delta)} = g'(\gamma)$, which is the optimal policy in Uzawa's paper [1969]: The price of capital is equal to the discounted stream of the returns from the "real" operations of the firm, where

Furthermore, the rate of gross capital formation is dependent only on the forward price of capital and independent of other variables. These results are a consequence of our assumption that the firm is a perfect competitor in the supply of new capital. Intuitively speaking, in each period the firm sells new capital at a price which is known with certainty. Therefore, it does not undertake any risks; it simply designs "ventures" and sells them to individual investors who bear the risks.[14]

The following corollary follows immediately from (2.2) and characterizes the optimal debt and equity policies in terms of the debt-equity ratio of Theorem 1. The following corollary characterizes the optimal equity and bond issues.

Corollary 1: Let $(x_t, \gamma_t, E_t, B_t)$ satisfy the conditions of Theorem 1. Then, for $\alpha_t > -1$,

$$E_t = \frac{p_t K_t (1+\gamma_t-\delta)}{(\hat{q}_t - x_t)(1+\alpha_t)} - S_t, \text{ and } B_t = p_t K_t (1+\gamma_t-\delta)\frac{\alpha_t}{1+\alpha_t}.$$

Next, it is shown that capital gains are positive if net capital formation takes place.

Corollary 2: $p_t(\gamma_t-\delta) - [g(\gamma_t)-g(\delta)] \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0$ if and only if $\gamma_t \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \delta$, $t = 0, 1, 2, \ldots$.

It is known that if $u(x_t)$ is homothetic, optimal policies are linear in the state variables. In the optimization problem (3.7), K_t, S_t are the state variables . In the modified problem (4.1)-(4.5) the state variable is $\hat{q}_t$

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the rate of discount is the cost of capital.

[14] Given the optimal investment policy of Theorem 1, $\hat{q}_t$ becomes effectively the state variable of the optimization problem (4.1)-(4.5). See (2.7) for clarification.

given the optimum γ_t. Theorem 2 relates the optimal financial policies to K_t and S_t for homothetic utility functions.

Proposition 1: If $u(kx)=\phi(k)u(kx)$, $\phi(k)>0$, $k>0$, and an optimal policy $(x_t^*, \gamma_t^*, E_t^*, B_t^*)$ exists, then

$$x_t^* = \theta_t \frac{K_t}{S_t}, \quad E_t = \sigma_t S_t, \quad B_t^* = \beta_t K_t^*, \quad t = 0,1,2,\ldots$$

for some constants $\theta_t>0$, $\sigma_t>0$, $\beta_t \geq 0$.

Thus, given past decisions, equity and bond issues are, respectively, linear functions of the outstanding number of shares and of the capital stock. It should be noted that stock buy-outs occur whenever the firm is a lender. This suggests that stock buy-outs are caused by poor real investment opportunities. The proof of Proposition 1 is given in the Appendix.

The optimization problem (4.1)-(4.3) is formally analogous to a dynamic portfolio problem. In particular, it is similar to the problem of Anastasopoulos, A. and Kounias, S. (1975). Several of their results, such as the conditions for the existence of equilibrium, can be proven for our case as well. This, and several generalizations of the results obtained here will be presented in a future paper.

APPENDIX

Proof of Lemma 3: It is sufficient to show that the upper (lower) support of the random variable defined by the expression $\frac{\hat{p}_{t+1}}{p_t} - r_{t+1}$ is positive (negative). To begin let $\bar{\bar{\gamma}}_{t+1}$ be the feasible policy which satisfies the condition $\bar{\bar{p}}_{t+1} = g'(\gamma_{t+1})$, $\bar{\bar{p}}_{t+1} \in A_{t+1}$. The upper support of this expression is:

$$\frac{1}{p_t}\left\{\bar{\bar{\rho}}_{t+1} + \bar{\bar{p}}_{t+1}\bar{\bar{\gamma}}_{t+1} - g(\bar{\bar{\gamma}}_{t+1}) + \bar{\bar{p}}_{t+1}(1-\delta)\right\} - \bar{r}_{t+1},$$

from (2.4),

$$> \frac{1}{p_t}\left\{\bar{\bar{\rho}}_{t+1} + \bar{\bar{p}}_{t+1}(1-\delta)\right\} - \bar{r}_{t+1} \geq 0,$$

by Assumptions 1 and 4. Furthermore, the lower support of $\frac{\hat{p}_{t+1}}{p_t} - r_{t+1}$ is

$$\frac{1}{p_t}\left\{\underline{\rho}_{t+1} - g(\delta) + 1\right\} - \bar{\bar{r}}_{t+1} < 0,$$

by Assumption 4.

Proof of Lemma 4: Let $u(x_t)$ be bounded above, thus the value of the objective function is finite. First we show that $x_t < \hat{q}_t$. Consider the following possibilities:

(i) $u'(0)=\infty$ and $u(0)=-\infty$. Then $x_t = 0$ for some t cannot be an optimal policy. Furthermore, $x_t = \hat{q}_t$ implies, from (3.1), $B_t=0$ and $K_{t+1}=0$, from (2.2) (i.e. the firm must be sold as a unit). Thus, by Assumption 8, $\hat{q}_{t+1}=x_{t+1}$, $k=1,2,\ldots$, which again cannot be an optimal policy.

(ii) $u'(0)=\infty$ and $u(0)$ finite. First we show that if $\hat{q}_t = x_t$ for some t, there exists another policy which is better. Without loss of generality let t=0. Then $x_o = \hat{q}_o$ implies, from (2.5), (2.2) and Assumption 8, $x_t=0$ for $t=1,2,\ldots$ In

this case the objective function becomes $u(\hat{q}_o)+ \frac{a}{1-a}\, u(0)$. Consider now the policy $\tilde{x}_o = \hat{q}_o - \varepsilon$, for some $\varepsilon>0$, $x_1=\varepsilon$, $\tilde{x}_t=0$ for $t=2,3,\ldots$ In the latter case the objective function becomes $u(\hat{q}_o - \varepsilon) + a\, u(\varepsilon) + \frac{a^2}{1+a}\, u(0)$. The difference of the objective functions is $u(\hat{q}_o - \varepsilon) - u(0) + a\left\{u(\varepsilon) - u(0)\right\} \geq u(\hat{q}_o - \varepsilon) - u(\hat{q}_0) + a\,[u(\varepsilon) - u(0)]$ which is positive for sufficiently small ε, by the assumption $u'(0)= \infty$ and the concavity of u.

Next we show that $x_t > 0$ for $t=0,1,2,\ldots$ (i) Let $x_1=0$ for all t. This cannot be optimal since it is dominated by the feasible policy $\tilde{\gamma}_t=\delta$, $\tilde{B}_t=0$, $S_o\tilde{x}_t = K_o[\rho_t - g(\delta)]$. (ii) Let $x_\tau=0$ for some τ. Without loss of generality $x_o = 0$ and $x_1>0$. Then the value of the objective function is $u(0) + aE[u(x_1)/H_o] + E[\sum_{t=2}^{\infty} a^t u(x_t)/H_o]$. Consider the feasible policy $\tilde{x}_o = \varepsilon$, $\tilde{x}_1 = x_1 - \varepsilon r_1$ for sufficiently small ε, and $\tilde{x}_t = x_t$ for $t\geq 2$. The value of the objective function for this policy is $u(\varepsilon) + aE[u(x_1-\varepsilon r_1)/H_o] + E\left[\sum_{t=2}^{\infty} a^t\, u(x_t)/H_o\right]$. The difference in the objective functions is $u(\varepsilon) - u(0) + a\,E[u(x_1-\varepsilon r_1)-u(x_1)/H_o]$ which is positive for sufficiently small ε, by the assumption $u'(0)=\infty$ and the concavity of u.

If $u(x_1)$ is not bounded above and bounded below, the proof that $\hat{q}_t - x_t > 0$ is the same as in the corresponding case (ii) above. The rest of the proof is trivial.

Proof of Theorem 1: Let $(x_t^*, \gamma_t^*, E_t^*, B_t^*)$ be the optimal policy of Theorem 1 and $(x_t, \gamma_t, E_t, B_t)$ be any other feasible policy, $t=0,1,2\ldots$. In the proof $\hat{p}_t^*$ and $\hat{q}_t^*$ refer to $\hat{p}_t$ and $\hat{q}_t$ which corresponds to γ_t^* and γ_t, respectively.

First we go through some preliminary steps. From (4.3) we obtain

(A.1) $\hat{p}_t^* - \hat{p}_t = p_t(\gamma_t^* - \gamma_t) - [g(\gamma_t^*) - g(\gamma_t)]$,

$\geq [p_t - g'(\gamma_t^*)](\gamma_t^* - \gamma_t)$, by Assumption 1.

Next, let $b_t \equiv \frac{B_t}{S_t + E_t}$ and consider the difference $(\hat{q}^*_{t+1} - \hat{q}_{t+1})$. From (2.5)

$P_t(\hat{q}^*_{t+1} - \hat{q}_{t+1}) = (\hat{q}_t^* - x_t^* + b_t^*)\hat{p}^*_{t+1} - (\hat{q}_t - x_t + b_t)\hat{p}_{t+1} - (b_t^* - b_t)p_t r_t$. Add and subtract $(\hat{q}_t - x_t - b_t)\,\hat{p}^*_{t+1}$ from the above expression:

(A.2) $P_t(\hat{q}^*_{t+1} - \hat{q}_{t+1}) = (\hat{q}_t^* - \hat{q}_t)\hat{p}^*_{t+1} - (x_t^* - x_t)\hat{p}^*_{t+1} + (\hat{p}^*_{t+1} - p_t r_t)(b_t^* - b_t)$

$+ (\hat{q}_t - x_t + b_t)\,(\hat{p}^*_{t+1} - \hat{p}_{t+1})$.

Next, we proceed with the proof. We will show that the policy of Theorem 1 overtakes any other feasible policy i.e.:

$E\left[\sum_{t=0}^{T} a^t\left\{u(x_t^*) - u(x_t)\right\}/H_o\right] \geq 0$, for $T >$ same T_o.

Note that

$E\left[\sum_{t=0}^{T-1} a^t\left\{u(x_t^*) - u(x_t)\right\}H_o\right] \geq E\left[\sum_{t=0}^{T-1} a^t u'(x_t^*)(x_t^* - x_t)/H_o\right]$,

by Assumption 2. Consider the expression $\sum_{t=0}^{T-1} a^t u'(x_t^*)(x_t^* - x_t)$. Solve (A.2) for $(x_t^* - x_t)$ and substitute the result in this expression:

$$\sum_{t=0}^{T-1} a^t u'(x_t^*)(x_t^* - x_t) = \sum_{t=0}^{T-1} a^t u'(x_t^*)\left\{(\hat{q}_t^* - \hat{q}_t) - (\hat{q}^*_{t+1} - \hat{q}_{t+1})\,\frac{p_t}{\hat{p}^*_{t+1}}\right.$$

$$\left. + \left[1 - \frac{p_t r_{t+1}}{\hat{p}^*_{t+1}}\right](b_t^* - b_t) + (\hat{q}_t - x_t + b_t)(\hat{p}^*_{t+1} - \hat{p}_{t+1})\,\frac{1}{\hat{p}^*_{t+1}}\right\}.$$

Change the order of summation:

$$= u'(x_o^*)(\hat{q}_o^* - \hat{q}_o) + \sum_{t=1}^{T-1} a^{t-1}(\hat{q}_t^* - \hat{q}_t)\left[au'(x_t^*) - \frac{p_{t-1}}{\hat{p}^*}\, u'(x_{t-1}^*)\right]$$

$$+ \sum_{t=1}^{T-1} a^{t-1}u'(x_{t-1}^*)(b_t^* - b_t)\left[1 - \frac{p_{t-1}r_t}{\hat{p}^*_t}\right] + \sum_{t=1}^{T-1} a^{t-1}u'(x_{t-1}^*)(\hat{q}_{t-1} - x_{t-1} + b_{t-1})\frac{1}{\hat{p}^*_t}$$

$$-a^{T-1}u'(x^{\circ}_{T-1})\left\{\frac{p_{T-1}}{\hat{p}^{\bullet}_{T}}\,(\hat{q}^{\bullet}_{T}-\hat{q}_{T})-(b^{\bullet}_{T-1}-b_{T-1})\left[1-\frac{p_{T-1}r_{T-1}}{\hat{p}^{\circ}_{T}}\right]-\frac{1}{\hat{p}^{\circ}_{T}}\,(\hat{q}_{T-1}-x_{T-1}+b_{T-1})(\hat{p}^{\bullet}_{T-1}-\hat{p}_{T-1})\right\}.$$

Substitute for $(\hat{q}^{\bullet}_{o}-\hat{q}_{o})$, $(\hat{q}^{\bullet}_{t}-\hat{q}_{t})$ and $(\hat{q}^{\bullet}_{T}-\hat{q}_{T})$ the expressions found from (A.2). After canceling common terms we have:

$$= u'(x^{\circ}_{o})\,\frac{K_{o}}{S_{o}}\,(\hat{p}^{\bullet}_{o}-\hat{p}_{o})+\sum_{t=1}^{T-1}a^{t-1}\left\{(\hat{q}^{\bullet}_{t-1}-\hat{q}_{t-1})-(x^{\bullet}_{t-1}-x_{t-1})\right\}\left[au'(x^{\bullet}_{t})\,\frac{\hat{p}^{\bullet}_{t}}{p_{t-1}}-u'(x^{\bullet}_{t-1})\right]+\sum_{t=1}^{T-1}u'(x^{\bullet}_{t})\left[\frac{\hat{p}^{\bullet}_{t}}{p_{t-1}}-r_{t}\right](b^{\bullet}_{t-1}-b_{t-1})+\sum_{t=1}^{T-1}a^{t}u'(x^{\bullet}_{t})(\hat{q}_{t-1}-x_{t-1}+b_{t-1})$$

$$(\hat{p}^{\bullet}_{t}-\hat{p}_{t})\,\frac{1}{p_{t-1}}-a^{T-1}u'(x^{\bullet}_{T-1})\left[(\hat{q}^{\bullet}_{T-1}-\hat{q}_{T-1})-(x^{\bullet}_{T-1}-x_{T-1})\right]$$

Take expectations with respect to $\rho_1,r_1,p_1,\rho_2,r_2,p_2,\ldots,\rho_{T-1},r_{T-1},p_{T-1}$:

$$(A.3) = u'(x^{\circ}_{o})\frac{K_{o}}{S_{o}}\,(\hat{p}^{\bullet}_{o}-\hat{p}_{o})+E\left[\sum_{t=1}^{T-1}a^{t-1}\left\{(\hat{q}^{\bullet}_{t-1}-x^{\bullet}_{t-1})-(\hat{q}_{t-1}-x_{t-1})\right\}E\left[au'(x^{\bullet}_{t})\,\frac{\hat{p}^{\bullet}_{t}}{p_{t-1}}-u'(x^{\bullet}_{t-1})/h_{t-1}\right]/H_{o}\right]+E\left[\sum_{t=1}^{T-1}a^{t}(b^{\bullet}_{t-1}-b_{t-1})E\left[u'(x^{\bullet}_{t})\left(\frac{\hat{p}^{\bullet}_{t}}{p_{t-1}}-r_{t}\right)/h_{t-1}\right]/H_{o}\right]$$

$$+E\left[\sum_{t=1}^{T-1}a^{t}\,\frac{1}{p_{t-1}}\,(\hat{q}_{t-1}-x_{t-1}+b_{t-1})\,E\left[u'(x_{t})(\hat{p}^{\bullet}_{t}-\hat{p}_{t})/h_{t-1}\right]/H_{o}\right]$$

$$-E\left[a^{T-1}u'(x^{\bullet}_{T-1})\left\{(\hat{q}^{\bullet}_{T-1}-x^{\bullet}_{T-1})-(\hat{q}_{T-1}-x_{T-1})\right\}/H_{o}\right].$$

In deriving (A.3) we have used the result: $E\left[\psi(h_t)/H_o\right] = E\left[E[\psi(h_t)/h_{t-1}]/H_o\right]$ $= E\left[E[\psi(h_t)/h_{t-1}]/H_o\right]$, which follows from Assumption 3.

If $x^{\circ}_{t}\,\epsilon(\bar{\alpha}_t,\bar{\bar{\alpha}}_t)$ conditions (i), (ii), (iii) of Theorem 1 and (A.1) imply:

$$E\left[au'(x^{\bullet}_{t})\,\frac{\hat{p}^{\bullet}_{t}}{p_{t-1}}-u'(x^{\bullet}_{t-1})/h_{t-1}\right]=E\left[u'(x^{\bullet}_{t})\left(\frac{\hat{p}^{\bullet}_{t}}{p_{t-1}}-r_{t-1}\right)/h_{t}\right]=0.$$

Furthermore, $(\hat{q}_{T-1}-x_{T-1}+x^{\circ}_{T-1})\geq 0$ by Lemma 4. Then (A.3) becomes:

$$(A.4)\quad \geq -E\left[a^{T-1}\,u'(x^{\bullet}_{T-1})\,\hat{q}^{\bullet}_{T-1}/H_{o}\right].$$

Suppose for some t $\alpha^{\bullet}_{t}=\bar{\alpha}_t$ or $\alpha^{\bullet}_{t}=\bar{\bar{\alpha}}_t$. Then,

$$(A.5)\quad \left\{(\hat{q}^{\bullet}_{t-1}-x^{\bullet}_{t-1})-(\hat{q}_{t-1}+x_{t-1})\right\}\bar{\alpha}_{t-1}\leq b^{\bullet}_{t-1}-b_{t-1}\leq\left\{(\hat{q}^{\bullet}_{t-1}-x^{\bullet}_{t-1})-(\hat{q}_{t-1}+x_{t-1})\right\}\bar{\bar{\alpha}}_{t-1}.$$

Substitute $\left\{(\hat{q}^{\bullet}_{t}-x^{\bullet}_{t})-(\hat{q}_{t}-x_{t})\right\}\alpha^{\circ}_{t-1}$ for $(b^{\bullet}_{t-1}-b_{t-1})$ in (A.3) and factor out common terms. By conditions (i), (iii), (A.1) and (A.5), (A.3) becomes

$$\geq E\left[a^{t-1}\left\{\hat{q}^{\bullet}_{t-1}-x^{\bullet}_{t-1})-(\hat{q}_{t-1}-x_{t-1}\right)E\left[au'(x^{\circ}_{t})\left\{\frac{\hat{p}^{\bullet}_{t}}{p_{t-1}}+\left(\frac{\hat{p}^{\bullet}_{t}}{p_{t-1}}-r_{t}\right)\alpha^{\circ}_{t-1}\right\}-u'(x^{\bullet}_{t-1})/h_{t-1}\right]/H_{o}\right]-E\left[a^{T-1}u'(x^{\bullet}_{T-1})\left\{(\hat{q}^{\bullet}_{T-1}-x^{\bullet}_{T-1})-(\hat{q}_{T-1}-x_{T-1})\right\}/H_{o}\right],$$

where the first term above is zero by condition (ii). Thus the above condition is

$$(A.4)\geq -\;E\left[a^{T-1}u'(x^{\bullet}_{T-1})\,\hat{q}^{\bullet}_{T-1}/H_{o}\right].$$

Finally take the limit of (A.4) as $t\to\infty$. If this limit is zero, it has been shown that the policy $(x^{\circ}_{t},\gamma^{\circ}_{t},E^{\bullet}_{t},B^{\bullet}_{t})$ overtakes any other feasible policy.

Proof of Corollary 2: The optimal policy (i) of Theorem 1 is continuous in p_t. Let $\tilde{p}_t$, $\tilde{p}_t>1$, be such that $\tilde{p}_t=g'(\delta)$. For $p_t=\tilde{p}_t$ it follows immediately that $p_t(\gamma_t-\delta)-[g(\gamma_t)-g(\delta)]=0$. We must show that for $p_t<\tilde{p}_t$ $(p_t>\tilde{p}_t)$ the above expression is negative (positive).

Without loss of generality, let $p_t=1$. Then the policy (i) of Theorem 1 yields $\gamma_t=0$, by Assumption 1. Substitute this result in the above expression to obtain $p_t(\gamma_t-\delta)-[g(\gamma_t)-g(\delta)] = -\delta+ g(\delta)<0$, again by Assumption 1. Finally, let $p_t> \tilde{p}_t$. The policy (i) of Theorem 1 gives $\gamma_t>\delta$. Furthermore, note that $p_t(\gamma_t-\delta)-[g(\gamma_t)-g(\delta)]\geq p_t(\gamma_t-\delta)-g'(\delta)(\gamma_t-\delta)= [p_t- g'(\delta)](\gamma_t-\delta) >0$, since $p_t> \tilde{p}_t=g'(\delta)$. The proof of the converse is trivial.

Proof of Proposition 1: To prove the proposition we first show that x_t^* and b_t^*, $b_t^* \equiv \frac{B_{t+1}}{S_{t+1}^*}$, are linear functions of $\hat{q}_t^*$, $t=0,1,2\ldots$.

Let (x_t,γ_t,E_t,B_t) be a feasible policy; i.e. one which satisfies (3.1) to (3.5). Consider the initial value of a share $\tilde{\tilde{q}}_o = k\hat{q}_o = k\frac{K_o}{S_o}\hat{p}_o$, for some k, $k>0$. Consider also the policy $\tilde{x}_t = kx_t$, $\tilde{\gamma}_t = \gamma_t$, $\tilde{b}_t = kb_t$ where $b_t \equiv \frac{B_t}{S_t+E_t}$. From (2.5) $\tilde{\tilde{q}}_{t+1} = \hat{kq}_{t+1}$. Thus, this policy is feasible when the initial value of a share is $\hat{kq}_0$.

Since x_t^* is optimal,

$$E\left[\sum a^t u(x_t^*)/H_o\right] \geq E\left[\sum a^t u(x_t)/H_o\right]$$

for every x_t feasible. Hence, when $\tilde{\tilde{q}}_o=\hat{kq}_o$

$$E\left[\sum a^t u(kx_t^*)/H_o\right] = \phi(k)E\left[\sum a^t u(x_t^*)/H_o\right]$$

$$\geq \phi(k)E\left[\sum a^t u(x_t)/H_o\right] = E\left[\sum a^t u(kx_t)/H_o\right] = E\left[\sum a^t u(\tilde{x}_t)/H_o\right]$$

for any feasible $\tilde{x}_t$. Thus, $\tilde{x}_t^* = kx_t^*$, $\tilde{b}_t^* = kb_t^*$. We know that $x_t=f(\hat{q}_t)$. Then, $\tilde{x}_t^* = f(\tilde{\hat{q}}_t^*) = f(\hat{kq}_t^*)$. But $\tilde{x}_t^* = kx_t^*$, so $f(\hat{kq}_t^*) = kf(\hat{q}_t^*)$ implies $f(\hat{q}_t^*)=\xi_t\hat{q}_t^*$ for some constant $\xi_t > 0$. Similar reasoning proves that $b_t^*=\psi_t\hat{q}_t^*$ for some constant $\psi_t\geq 0$.

Next, the linearity of x_t^* and b_t^* in $\hat{q}_t^*$ and (2.2) imply that

$$(A.6)\quad \hat{q}_t^* = \frac{p_t}{1-\xi_t+\psi_t}\frac{K_{t+1}^*}{S_{t+1}^*} >0.$$

From the definition of b_t and (A.6) we obtain,

$$B_t = \psi_t\hat{q}_t^*S_{t+1}^* = \frac{\psi_t}{1-\xi_t+\psi_t}p_t(1+\gamma_t-\delta)K_t \equiv \eta_t\psi_t p_t(1+\gamma_t-\delta)K_t \equiv \beta_t K_t ,$$

where, $\eta \equiv \frac{\psi_t}{1-\xi_t+\psi_t}$

Lagging the above expression by one period and combining it with (2.1) yields

$$(A.7)\quad \hat{q}_t = \hat{p}_t\frac{K_t}{S_t} - \frac{B_{t-1}r_t}{S_t} = \hat{p}_t\frac{K_t}{S_t} + \psi_{t-1}\hat{q}_{t-1}^*r_t = \left[\hat{p}_t - \frac{\psi_{t-1}p_{t-1}r_t}{1-\xi_{t-1}+\psi_{t-1}}\right]\frac{K_t}{S_t}$$

$$= \left[\hat{p}_t - \eta_{t-1}p_{t-1}r_t\right]\frac{K_t}{S_t} .$$

Expressions (A.7) and (A.6) yield:

$$E_t^* = \left\{\frac{p_t(1+\gamma_t-\delta)}{(1-\xi_t+\psi_t)\left[\hat{p}_t - \eta_{t-1}p_{t-1}r_t\right]} - 1\right\}S_t \equiv \sigma_t S_t > 0.$$

Finally, θ_t is defined as follows: $\theta_t = \left[\hat{p}_t - \eta_{t-1}p_{t-1}r_t\right]\xi_t > 0$. This concludes the proof.

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